

# Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions

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## Abstract

Euclidean gravity method has been successful in computing logarithmic corrections to extremal black hole entropy in terms of low energy data, and gives results in perfect agreement with the microscopic results in string theory. Motivated by this success we apply Euclidean gravity to compute logarithmic corrections to the entropy of various non-extremal black holes in different dimensions, taking special care of integration over the zero modes and keeping track of the ensemble in which the computation is done. These results provide strong constraint on any ultraviolet completion of the theory if the latter is able to give an independent computation of the entropy of non-extremal black holes from microscopic description. For Schwarzschild black holes in four space-time dimensions the macroscopic result seems to disagree with the existing result in loop quantum gravity.

# Contents

<b>1</b>	<b>Introduction and summary</b>	<b>2</b>
<b>2</b>	<b>Grand canonical partition function</b>	<b>8</b>
2.1	General framework . . . . .	8
2.2	Heat kernel . . . . .	10
2.3	One loop contribution to the partition function from the non-zero modes . . .	11
2.4	One loop contribution to the partition function from the zero modes . . . . .	18
2.5	Higher loop contributions . . . . .	20
2.6	Effect of cosmological constant term . . . . .	21
<b>3</b>	<b>Black hole entropy in microcanonical and other ensembles</b>	<b>22</b>
3.1	Entropy in the microcanonical ensemble . . . . .	22
3.2	Entropy in other ensembles . . . . .	25
<b>4</b>	<b>Comparison with loop quantum gravity prediction</b>	<b>27</b>
<b>5</b>	<b>BTZ black holes</b>	<b>29</b>
5.1	Microscopic computation . . . . .	30
5.2	Macroscopic computation . . . . .	31
5.3	Comparison of the microscopic and the macroscopic results . . . . .	32
<b>A</b>	<b>Normalizability of the zero modes</b>	<b>33</b>
<b>B</b>	<b>Subtraction of the thermal gas contribution to the partition function</b>	<b>35</b>

## 1 Introduction and summary

One of the tests of any theory of quantum gravity is a successful comparison between the macroscopic and the microscopic prediction of black hole entropy. This also provides us with a deep connection between the infrared and the ultraviolet properties of gravity. The leading contribution to the black hole entropy, given by the Bekenstein-Hawking formula, can be computed from the low energy properties of gravity; yet it must agree with the logarithm of the microstate degeneracy which is sensitive to the ultraviolet completion of the theory.

Another property of the black hole entropy that can be computed from the knowledge of the infrared physics is the logarithmic correction to the black hole entropy. By taking appropriate scaling limit of the mass, charge and other quantum numbers carried by the black hole one can ensure that the size of the black hole becomes large, but other dimensionless ratios remain fixed. In this case the dominant contribution to the entropy comes from the Bekenstein-Hawking term, but it can receive subleading corrections proportional to the logarithm of the horizon area [1–17]. On the macroscopic side these corrections arise from one loop contribution to the black hole partition function. Computation of the full one loop contribution would certainly require knowledge of the ultraviolet completion of the theory, but the logarithmic corrections arise only from loops of massless fields and from the range of loop momentum integration where the loop momenta remain much smaller than the Planck scale. Thus this can be evaluated purely from the knowledge of the low energy data – the spectrum of massless fields and their coupling to the black hole background. Requiring the microscopic counting results to agree with this would give strong constraint on any proposal for the ultraviolet completion of gravity.

Recently Euclidean gravity approach has been used to compute the logarithmic corrections to the entropy of a certain class of extremal black holes in string theory [18–22]. Whenever the corresponding microscopic results are available – *e.g.* for BPS black hole entropy in four dimensional  $\mathcal{N} = 4$  and  $\mathcal{N} = 8$  supersymmetric theories, and BMPV black hole [23,24] entropy in five dimensional string theory – these macroscopic results are in perfect agreement with the microscopic results [18, 19, 22]. Macroscopic results are also available for BPS black holes in  $\mathcal{N} = 2, 3, 5$  and 6 supersymmetric theories in four dimensions [20,21] but concrete microscopic results are not yet available. A summary of the current results on the logarithmic corrections to the entropy of extremal supersymmetric black holes can be found in [25]. For the benefit of the reader we have reproduced this in table 1. Macroscopic results also exist for a certain class of extremal non-supersymmetric black holes [20,22,26] but there are no microscopic results to compare them with.

Motivated by this success, in this paper we use the Euclidean gravity approach to compute logarithmic corrections to the entropy of non-extremal black holes. This has been done before using many different approaches. Our approach is most closely related to the one due to Solodukhin, Fursaev and others [1–3] (see *e.g.* [27] for a review). The main difference between our approach and those reviewed in [27] is threefold: (i) we take into account possible contribution to the conformal anomaly due to the presence of background fields other than the gravitational field, (ii) we give special treatment to integration over the zero modes and (iii) we keep track

The theory	scaling of charges	logarithmic contribution	microscopic
$\mathcal{N} = 4$ supersymmetric CHL models in $D = 4$ and type II on $K3 \times T^2$ with $n_v$ matter multiplet	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	0	$\checkmark$
Type II on $T^6$	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	$-8 \ln \Lambda$	$\checkmark$
$\mathcal{N} = 2$ supersymmetric theories in $D = 4$ with $n_V$ vector and $n_H$ hyper multiplets	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	$\frac{1}{6}(23 + n_H - n_V) \ln \Lambda$	?
$\mathcal{N} = 6$ supersymmetric theories in $D = 4$	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	$-4 \ln \Lambda$	?
$\mathcal{N} = 5$ supersymmetric theories in $D = 4$	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	$-2 \ln \Lambda$	?
$\mathcal{N} = 3$ supersymmetric theories in $D = 4$ with $n_v$ matter multiplets	$Q_i \sim \Lambda, \quad A \sim \Lambda^2$	$2 \ln \Lambda$	?
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with $n_V$ vectors preserving 16 or 32 supercharges	$Q_1, Q_5, n \sim \Lambda,$ $J = 0, \quad A \sim \Lambda^{3/2}$	$-\frac{1}{4}(n_V + 3) \ln \Lambda$	$\checkmark$
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with $n_V$ vectors preserving 16 or 32 supercharges	$Q_1, Q_5, n \sim \Lambda,$ $J \sim \Lambda^{3/2}, \quad A \sim \Lambda^{3/2}$	$-\frac{1}{4}(n_V - 3) \ln \Lambda$	$\checkmark$

Table 1: Macroscopic predictions for the logarithmic corrections to extremal black hole entropy in different string theories and the status of their comparison with the microscopic results. The first column describes the theory and the black hole under consideration. The second column describes the scaling of the various charges, as well as the area  $A$  of the event horizon, under which the logarithmic correction is computed. For all four dimensional theories,  $Q_i$  in the second column stands for all the electric and magnetic charges of the black hole. For BMPV black holes in five space-time dimensions,  $Q_1$ ,  $Q_5$ ,  $n$  and  $J$  stand respectively for the D1-brane charge, D5-brane charge, Kaluza-Klein momentum and the angular momentum (under the  $SU(2)_L$  subgroup of the rotation group). The third column describes the macroscopic results for the logarithmic correction to the entropy under the scaling described in the second column. In the last column a  $\checkmark$  indicates that the microscopic results are available and agree with the macroscopic prediction while a ? indicates that the microscopic results are not yet available.

of any additional logs which may be generated while converting the result on the partition function to the result on entropy via a Laplace transform. As our experience with extremal black holes show, all these effects are important and only after including these effects we can get agreement between the microscopic and the macroscopic results. Another technical aspect of our analysis is that unlike in the approach reviewed in [27], we do not need to study quantization of fields in a space-time with conical defect. Instead we need to compute the partition function of various fields in the euclidean black hole space-time and interpret the result as the grand canonical partition function of the black hole. Although this is not expected to affect the final result [1, 28], not having to deal with background with conical defects is particularly important in the context of string theory where the procedure for quantizing strings in the presence of conical defect with arbitrary defect angle is not completely understood.

Unfortunately there are no concrete microscopic counting results for non-extremal black holes in string theory, and so we cannot at present use the results of this paper to test string theory. There are however computations of logarithmic corrections to Schwarzschild black hole entropy in loop quantum gravity [29–41]. We compare our result with these results and find disagreement. In particular contribution to the logarithmic correction from the massless graviton loop seems to be missing from the loop quantum gravity results. This could be related to the difficulty in obtaining semiclassical limit in loop quantum gravity. The other case where microscopic results for non-extremal black holes are available – BTZ black holes [7] – the euclidean gravity prediction and the microscopic results agree trivially [8, 42, 43]. This will be reviewed in §5.

We shall now summarize our main results. We consider rotating black hole solution in  $D$  dimensional space-time carrying generic angular momentum so that the symmetry group of the black hole solution is generated by the Cartan subalgebra of the rotation group. We define the microcanonical entropy  $S_{\text{mc}}$  to be such that  $e^{S_{\text{mc}}}$  gives the number of quantum states of the black hole per unit interval of mass, carrying fixed angular momentum and charges.<sup>1</sup> While fixing the angular momentum we fix only the components along the Cartan generators (*e.g.*  $J_3$  for the  $SO(3)$  rotation group), but sum over all values of the Casimirs (*e.g.*  $\sum_{i=1}^3 J_i J_i$  for  $SO(3)$  rotation group). In this case  $S_{\text{mc}}$ , expressed as a function of the mass  $M$ , angular

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<sup>1</sup>Throughout this paper we shall work in  $\hbar = c = G_N = 1$  units.

momenta  $\vec{J}$  along the Cartan generators<sup>2</sup> and charges  $\vec{Q}$  receives a logarithmic correction:

$$S_{\text{mc}}(M, \vec{J}, \vec{Q}) = S_{\text{BH}}(M, \vec{J}, \vec{Q}) + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-2)N_C - \frac{1}{2}(D-4)n_V \right), \quad (1.1)$$

where  $S_{\text{BH}}(M, \vec{J}, \vec{Q})$  is the classical Bekenstein-Hawking entropy of a black holes carrying mass  $M$ , angular momenta  $\vec{J}$  and charges  $\vec{Q}$ ,  $a$  is the black hole size parameter, related to the horizon area  $A_H$  via  $A_H \sim a^{D-2}$ ,  $N_C = [(D-1)/2]$  is the number of Cartan generators of the rotation group and  $n_V$  is the number of  $U(1)$  gauge fields.  $C_{\text{local}}$  is related to the contribution to the trace anomaly due to the massless fields in the black hole background [44–52]. In any given theory this can in principle be computed using the procedure given *e.g.* in [51]. In particular  $C_{\text{local}}$  vanishes in odd dimensions, and for uncharged black holes in  $D = 4$  we have (see *e.g.* [27])

$$C_{\text{local}} = \frac{1}{90} \left( 2n_S - 26n_V + 7n_F - \frac{233}{2}n_{3/2} + 424 \right), \quad (1.2)$$

if the theory contains, besides gravity,  $n_S$  massless scalar fields,  $n_V$  massless vector fields,  $n_F$  massless Dirac fields and  $n_{3/2}$  massless spin 3/2 fields, all minimally coupled to gravity without any other interactions. The last term 424 is the contribution from the graviton loop. In pure gravity theory only this term is present. In  $D = 4$  our result (1.1) differs from the earlier result given *e.g.* in [3] by the  $-N_C \ln a = -\ln a$  term that arises while converting the result on partition function to the result on entropy. The expression for  $C_{\text{local}}$  for a Kerr-Newmann black hole in four space-time dimensions can be computed using the recent results on the heat kernel expansion in Einstein-Maxwell theory [26] and the result has been given in eq.(2.34).

We can also consider the entropy in a mixed ensemble, defined as

$$e^{S_{\text{mixed}}(M, \vec{Q})} = \sum_{\vec{J}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})}, \quad (1.3)$$

where the sum runs over all eigenvalues of the Cartan generators of the rotation algebra. Thus  $e^{S_{\text{mixed}}(M, \vec{Q})} \delta M$  counts the total number of states of all angular momentum in the mass range  $(M, M + \delta M)$  and fixed values of the charges, with the leading contribution coming from near zero angular momentum states where  $S_{\text{mc}}$  is maximum. The euclidean gravity analysis leads to the following prediction for the logarithmic correction to  $S_{\text{mixed}}$ :

$$S_{\text{mixed}}(M, \vec{Q}) = S_{\text{BH}}(M, \vec{J} = 0, \vec{Q}) + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-4)n_V \right). \quad (1.4)$$

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<sup>2</sup>Here the vector sign on  $\vec{J}$  stands not for all components of the angular momentum but only the Cartan generators. Thus for example for the  $\text{SO}(3)$  rotation group  $\vec{J}$  is a one component vector labelling  $J_3$ .

Finally we can consider the entropy in the ensemble containing only singlet states of the rotation group. If we denote the corresponding entropy by  $S_{\text{singlet}}$  then  $\exp[S_{\text{singlet}}(M, \vec{Q})]\delta M$  counts the total number of rotationally invariant states in the mass interval  $\delta M$  and charge  $\vec{Q}$ . Our result for  $S_{\text{singlet}}(M, \vec{Q})$  takes the form:

$$S_{\text{singlet}}(M, \vec{Q}) = S_{\text{BH}}(M, \vec{J} = 0, \vec{Q}) + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-2)N_R - \frac{1}{2}(D-4)n_V \right), \quad (1.5)$$

where  $N_R = (D-1)(D-2)/2$  is the total number of generators of the rotation group.

In a theory of pure gravity in  $D = 4$ , (1.2) gives  $C_{\text{local}} = 212/45$ . Also we have  $N_R = 3$ . Hence the logarithmic correction to  $S_{\text{singlet}}$  is given by  $\Delta S_{\text{singlet}} = \left(\frac{212}{45} - 3\right) \ln a$ . In contrast we find that loop quantum gravity result of [29–41] predicts a result of  $-2 \ln a$ .<sup>3</sup> The two results obviously disagree.

Since on the macroscopic side the computation is based on one loop determinant of massless fields in the black hole background, one could ask whether higher loops can give additional logarithmic corrections to the black hole entropy. We have checked, based on naive power counting arguments, that higher loops do not give any logarithmic corrections to the entropy. This is related to the fact that in a theory of gravity infrared divergences become softer at higher loops. It has also been argued in [18] that neither massive fields nor higher derivative corrections to the action can affect the logarithmic correction to the entropy. Thus the macroscopic results for logarithmic corrections seem quite robust.

The rest of the paper is organized as follows. In §2 we describe the computation of logarithmic corrections to the grand canonical partition functions using Euclidean gravity path integral. In §3 we discuss how to translate these results into a statement of logarithmic corrections to the black hole entropy in different ensembles. In §4 we compare the macroscopic prediction for the logarithmic correction to the entropy of a Schwarzschild black hole to the prediction of loop quantum gravity. Finally in §5 we review how euclidean gravity approach can be used to correctly reproduce the logarithmic corrections to the entropy of a BTZ black hole. Appendix A contains an analysis of the zero modes of the euclidean black hole solution, and in appendix B we describe the procedure for removing from the partition function the contribution due to the thermal gas around the black hole.

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<sup>3</sup>For example the result of [30] gives a logarithmic correction of  $-\frac{3}{2} \ln A_H = -3 \ln a$  for the entropy of singlet states of the rotation group. But this counts the number of states per unit interval in the area variable  $A_H$ . Converting this to number of states per unit interval in mass gives a result of  $-2 \ln a$ .

## 2 Grand canonical partition function

In this section we shall compute logarithmic corrections to black hole partition function in Einstein's theory of gravity in  $D$  dimensions coupled to a set of massless abelian vector fields  $\{A_\mu^{(\alpha)}\}$ , a set of other massless neutral scalar fields  $\{\varphi_s\}$  and also possibly neutral Dirac and Rarita-Schwinger fields. We shall assume that the Lagrangian density  $\mathcal{L}$  has a scaling property so that purely bosonic terms all have two derivatives, all terms with two fermion fields have a single derivative and all terms with four fermion fields have no derivatives. This covers a wide range of theories *e.g.* pure gravity, pure gravity coupled to Maxwell fields, scalars, fermions and other fields via minimal coupling with no other interaction between these other fields, a variety of extended supergravity theories at generic point in the moduli space of these theories etc. This however excludes theories with cosmological constant – we shall comment on them briefly at the end of this section and again in §5 where we discuss the case of BTZ black holes.

### 2.1 General framework

Due to the absence of cosmological constant the Minkowski space is a solution of this theory, and we can consider a charged, rotating black hole solution which asymptotes to Minkowski space. We shall denote by  $t$  the time coordinate, by  $x^i$  for  $1 \leq i \leq (D-1)$  the spatial coordinates, and by  $M$ ,  $Q_\alpha$  and  $J_{ij}$  the mass, electric charges and angular momenta carried by the black hole.<sup>4</sup> As a consequence of the scaling symmetry mentioned above, given any classical black hole solution we can generate a whole family of solutions by a scaling

$$g_{\mu\nu} \rightarrow \Lambda^2 g_{\mu\nu}, \quad A_\mu^{(\alpha)} \rightarrow \Lambda A_\mu^{(\alpha)}, \quad \varphi_s \rightarrow \varphi_s, \quad (2.1)$$

and under such a scaling the classical action as well as the Bekenstein-Hawking entropy scales by  $\Lambda^{D-2}$ , the mass and the electric and magnetic charges scale by  $\Lambda^{D-3}$ , and the angular momentum scales by  $\Lambda^{D-2}$ . This leads to the relations

$$\begin{aligned} S_{\text{BH}}(\Lambda^{D-3}M, \Lambda^{D-2}\vec{J}, \Lambda^{D-3}\vec{Q}) &= \Lambda^{D-2} S_{\text{BH}}(M, \vec{J}, \vec{Q}), \\ a(\Lambda^{D-3}M, \Lambda^{D-2}\vec{J}, \Lambda^{D-3}\vec{Q}) &= \Lambda a(M, \vec{J}, \vec{Q}), \end{aligned} \quad (2.2)$$

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<sup>4</sup>A black hole can also carry magnetic charges associated with gauge fields in four dimensions. More generally in  $D$  space-time dimensions a black hole can carry magnetic charges of  $(D-3)$ -form gauge fields. For simplicity of notation we shall not explicitly display the dependence of various quantities on these magnetic charges, but it should be understood that one can easily generalize the scaling laws described in (2.2) if such charges are present.



where  $a$  is the length parameter that gives the size of the black hole, *e.g.* the area of the event horizon scales as  $a^{D-2}$ .

As is well known, when we analytically continue the black hole solution to Euclidean space-time, the time coordinate needs to be periodically identified with period given by the inverse temperature  $\beta$ . The other parameters labelling the Euclidean solution are the chemical potentials  $\mu_\alpha$  dual to the electric charges  $Q_\alpha$  and the angular velocities  $\omega_{ij}$  dual to the angular momenta  $J_{ij}$ . Physically  $\mu_\alpha/\beta$  parametrize the component of the gauge field  $A_\mu^{(\alpha)}$  along the asymptotic Euclidean time circle, and  $\omega_{ij}/\beta$  label the asymptotic values of  $g_{t\phi_{ij}}$  with  $\phi_{ij}$  denoting the angle of rotation in the  $x^i$ - $x^j$  plane. For the classical black hole  $\beta$ ,  $\mu_\alpha$  and  $\omega_{ij}$  are determined in terms of  $M$ ,  $Q_\alpha$  and  $J_{ij}$  and vice versa. Without any loss of generality we can take  $\omega_{ij}$  to be of the form  $\omega_{ij} = \sum_a \omega_a T_{ij}^a$  where  $T_{ij}^a$  are the Cartan generators and  $\omega^a$ 's are constants. Thus  $\{\omega^a\}$  describe a vector of dimension  $N_C$  where  $N_C$  is the rank of the rotation group.

In order to calculate the quantum corrections to the black hole entropy we shall follow the Euclidean path integral approach since this has successfully reproduced the correct results for extremal black hole entropy in many cases [18, 19, 22]. The euclidean partition function is defined as [53]

$$Z(\beta, \vec{\omega}, \vec{\mu}) = \int [D\Psi] e^{-\mathcal{S}_E(\Psi)} \quad (2.3)$$

where  $\Psi$  stands for all the fields in the theory including the metric and the gauge fields,  $\mathcal{S}_E(\Psi)$  is the Euclidean action and the path integral is performed subject to the asymptotic boundary conditions set by the parameters  $\beta$ ,  $\vec{\omega}$  and  $\vec{\mu}$ . While  $\beta$  labels the period of the euclidean time coordinate,  $\mu_\alpha/\beta$  denote the component of the asymptotic gauge field  $A_\mu^{(\alpha)}$  along the euclidean time, and  $\omega^a/\beta$ 's denote the  $t$ - $\phi^a$  components of the asymptotic metric where  $\phi^a$  is the angular coordinate conjugate to the Cartan generator  $T^a$ . Note that in the boundary conditions defining the path integral there is no reference to a black hole solution; the black hole becomes relevant as a saddle point which contributes to the path integral.<sup>5</sup> Furthermore, although at the saddle point  $M$ ,  $Q_\alpha$  and  $J_{ij}$  are determined in terms of  $\beta$ ,  $\mu_\alpha$  and  $\omega_a$ , while carrying out the path integral we only keep fixed  $\beta$ ,  $\mu_\alpha$  and  $\omega_a$ , and allow fluctuations carrying different values of  $M$ ,  $Q_\alpha$  and  $J_{ij}$ .

Classical contribution  $Z_{\text{cl}}$  to  $Z$  is given by  $\exp[-\mathcal{S}_E(\Psi_{\text{cl}})]$  where  $\Psi_{\text{cl}}$  denotes the classical black hole solution. This is related to the classical Bekenstein-Hawking entropy  $S_{\text{BH}}(M, \vec{J}, \vec{Q})$

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<sup>5</sup>Note also that the Euclidean black hole solution is complex, but this will not affect our analysis since we shall evaluate the path integral in the saddle point approximation.

by a Legendre transform [53]:

$$\begin{aligned}
S_{\text{BH}}(M, \vec{J}, \vec{Q}) &= \ln Z_{\text{cl}}(\beta, \vec{\omega}, \vec{\mu}) + \beta M + \omega^a J_a + \mu_\alpha Q_\alpha, \\
\beta &= \frac{\partial S_{\text{BH}}}{\partial M}, \quad \omega_b = \frac{\partial S_{\text{BH}}}{\partial J_b}, \quad \mu_\alpha = \frac{\partial S_{\text{BH}}}{\partial Q_\alpha} \\
\Leftrightarrow \quad M &= -\frac{\partial \ln Z_{\text{cl}}}{\partial \beta}, \quad J_b = -\frac{\partial \ln Z_{\text{cl}}}{\partial \omega_b}, \quad Q_\alpha = -\frac{\partial \ln Z_{\text{cl}}}{\partial \mu_\alpha},
\end{aligned} \tag{2.4}$$

where  $\{J_a\}$  are related to the angular momenta  $\{J_{ij}\}$  via  $J_{ij} = \sum_a J_a T_{ij}^a$ . From (2.4), (2.2) it follows that

$$\ln Z_{\text{cl}}(\Lambda\beta, \vec{\omega}, \Lambda\vec{\mu}) = \Lambda^{D-2} \ln Z_{\text{cl}}(\beta, \vec{\omega}, \vec{\mu}). \tag{2.5}$$

## 2.2 Heat kernel

Our goal in this subsection is to compute one loop quantum correction to  $\ln Z$  in the limit of large  $\Lambda$  and extract corrections to  $\ln Z$  of order  $\ln \Lambda$  or equivalently  $\ln a$ . This is done using the heat kernel technique which we shall now describe.

Let  $\{\phi^\ell\}$  denote the set of massless fields in the theory. Here the index  $\ell$  labels different scalar fields as well as the space-time indices of tensor fields. Let  $\{f_n^{(\ell)}(x)\}$  denote an orthonormal basis of eigenfunctions of the kinetic operator expanded around the near horizon background, with eigenvalues  $\{\kappa_n\}$ . The orthonormality relations take the form:

$$\int d^D x \sqrt{\det g} G_{\ell\ell'} f_n^{(\ell)}(x) f_m^{(\ell')}(x) = \delta_{mn}, \tag{2.6}$$

where  $g_{\mu\nu}$  is the metric of the Euclidean black hole space time and  $G_{\ell\ell'}$  is a metric in the space of fields induced by the background metric, *e.g.* for a vector field  $A_\mu$ ,  $G^{\mu\nu} = g^{\mu\nu}$ . The heat kernel  $K^{\ell\ell'}(x, x')$  is defined as

$$K^{\ell\ell'}(x, x'; s) = \sum_n e^{-\kappa_n s} f_n^{(\ell)}(x) f_n^{(\ell')}(x'). \tag{2.7}$$

Among the  $f_n^{(\ell)}$ 's there may be a special set of modes for which  $\kappa_n$  vanishes. We call these zero modes and define

$$K'^{\ell\ell'}(x, x'; s) = \sum'_n e^{-\kappa_n s} f_n^{(\ell)}(x) f_n^{(\ell')}(x'), \tag{2.8}$$

where the prime on the sum denotes that we remove the contribution from the zero modes. We also define

$$K(x; s) = G_{\ell\ell'} K^{\ell\ell'}(x, x; s), \quad K'(x; s) = G_{\ell\ell'} K'^{\ell\ell'}(x, x; s). \tag{2.9}$$

Using (2.6)-(2.9) we now get

$$\int d^D x \sqrt{\det g} K'(x; s) = \sum_n' e^{-\kappa_n s}, \quad (2.10)$$

$$\int d^D x \sqrt{\det g} K(x; s) = \int d^D x \sqrt{\det g} K'(x; s) + N_{\text{zm}}, \quad (2.11)$$

where  $N_{\text{zm}}$  denotes the number of zero modes.

If there are fermion fields present then the definitions of  $K$  and  $K'$  are modified in two ways. First of all since the fermion kinetic term is linear in derivatives, we take  $\kappa_n$ 's to be the eigenvalues of the square of the fermion kinetic operator and insert a factor of  $1/2$  in the trace in (2.9). Second for fermion fields we insert an extra minus sign into the trace in (2.9).

If we denote by  $Z_{\text{nz}}$  and  $Z_{\text{zm}}$  the one loop contribution to the partition function from integration over the non-zero modes and the zero modes respectively, then the net result for the partition function to one loop can be expressed as

$$\ln Z = \ln Z_{\text{cl}} + \ln Z_{\text{nz}} + \ln Z_{\text{zm}}. \quad (2.12)$$

We shall now discuss the evaluation of  $Z_{\text{nz}}$  and  $Z_{\text{zm}}$ .

### 2.3 One loop contribution to the partition function from the non-zero modes

The one loop contribution to  $\ln Z$  from the non-zero modes is given by

$$-\frac{1}{2} \sum_n' \ln \kappa_n, \quad (2.13)$$

with the understanding that for fermions there is an additional factor of  $-1/2$  multiplying the summand. (2.13) of course has many divergences – both ultraviolet and infrared – and to extract something meaningful we need to understand the role of these divergences. First of all we must remember that  $Z(\beta, \vec{\omega}, \vec{\mu})$  defined in (2.3) describes a grand canonical partition function, and the contribution to (2.3) from the saddle point corresponding to the Euclidean black hole solution can be interpreted as due to a black hole in equilibrium with a thermal gas of all the massless (and massive) particles in the theory. In the infinite volume limit the contribution due to the thermal gas is infinite; so we need to first regularize this by confining the black hole in some box of size  $L$ , putting appropriate boundary condition on

all the fields at the boundary of the box. The leading contribution to  $\ln Z$  from the thermal gas is now given by  $L^{D-1}f(\beta, \vec{\omega}, \vec{\mu})$  where  $f$  is some function that scales as  $\Lambda^{-D+1}$  under the scaling (2.2), (2.5). There are also possible subleading contributions involving lower positive powers of  $L$  due to boundary effects. We must subtract all these contributions from  $\ln Z$  in order to identify the contribution to the partition function associated with the black hole microstates. To do this we introduce a length  $a_0$  which is fixed but large compared to the Planck length, consider another black hole solution which is related to the original solution by a rescaling of the parameters described in (2.2), (2.5) with  $\Lambda = a_0/a$  and confine this new system in a box of side  $L_0 = La_0/a$ . In the common coordinate system in which the metric for the new and the original black hole solutions are simply related by a multiplicative factor of  $(a_0/a)^2$ , the shape of the box in which we confine the two black holes are taken to be identical, and furthermore the boundary conditions on the various fields are taken to be related by the scale transformation (2.1) with  $\Lambda = a_0/a$ . Let  $Z_0$  be the partition function of the new black hole solution. The leading contribution to  $\ln Z_0$  from the thermal gas is given by  $(La_0/a)^{D-1}f(a_0\beta/a, \vec{\omega}, a_0\vec{\mu}/a) = L^{D-1}f(\beta, \vec{\omega}, \vec{\mu})$ . This is identical to the leading thermal gas contribution to  $\ln Z$  and hence subtracting  $\ln Z_0$  from  $\ln Z$  has the effect of removing the leading contribution to  $\ln Z$  due to the thermal gas. In fact we have argued in appendix B that subtracting  $\ln Z_0$  from  $\ln Z$  also removes the spurious boundary terms, proportional to subleading powers of  $L$ , which may be present. On the other hand since the new black hole has a fixed size  $a_0$ , subtracting  $\ln Z_0$  does not remove the  $a$  dependent contribution to  $\ln Z$  that comes from the intrinsic entropy of the black hole of size  $a$ . By an abuse of notation we shall continue to denote the regulated partition function  $Z/Z_0$  by the same symbol  $Z$ . If we denote by  $\kappa_n^{(0)}$  the eigenvalues of the kinetic operator of the new black hole solution then the one loop contribution to  $\ln Z$  from the non-zero modes after the subtraction is given by

$$\ln Z_{\text{nz}} = -\frac{1}{2} \sum_n' (\ln \kappa_n - \ln \kappa_n^{(0)}). \quad (2.14)$$

We shall compute (2.14) using Schwinger proper time formalism. We use the relation

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\infty} \frac{ds}{s} (e^{-As} - e^{-Bs}) = \ln \frac{B}{A}, \quad (2.15)$$

to express (2.14) as

$$\ln Z_{\text{nz}} = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_n' (e^{-\kappa_n s} - e^{-\kappa_n^{(0)} s}), \quad (2.16)$$

where  $\epsilon$  is an  $a$ -independent ultraviolet cut-off. In an ultraviolet regulated theory  $\epsilon$  is of the order of Planck length<sup>2</sup> which in our convention is of order unity. The final result will not depend on the details of this cut-off.

To proceed further we note that the two black hole solutions of size  $a$  and  $a_0$  as well as their infrared cut-offs  $L$  and  $La_0/a$  are related by a rescaling of the metric and the gauge fields as given in (2.1) with  $\Lambda = a_0/a$ . It follows from this that the eigenvalues  $\kappa_n$  and  $\kappa_n^{(0)}$  are related as

$$\kappa_n^{(0)} = \kappa_n a^2 / a_0^2. \quad (2.17)$$

Hence (2.16) can be written as

$$\ln Z_{\text{nz}} = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_n' e^{-\kappa_n s} - \frac{1}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_n' e^{-\kappa_n s a^2 / a_0^2} = \frac{1}{2} \int_{\epsilon}^{\epsilon a^2 / a_0^2} \frac{ds}{s} \sum_n' e^{-\kappa_n s}, \quad (2.18)$$

where in the last step we have carried out a rescaling of variable  $s a^2 / a_0^2 \rightarrow s$  in the second term. Using (2.10), (2.11) we can express (2.18) as

$$\ln Z_{\text{nz}} = \frac{1}{2} \int_{\epsilon}^{\epsilon a^2 / a_0^2} \frac{ds}{s} \left( \int d^D x \sqrt{\det g} K(x; s) - N_{\text{zm}} \right). \quad (2.19)$$

From (2.19) we see that the variable  $s/a^2$  ranges between  $\epsilon/a^2$  and  $\epsilon/a_0^2$ , and hence remains small over the entire integration range since both  $a$  and  $a_0$  are taken to be large compared to the length cut-off  $\sqrt{\epsilon}$ . This allows us to use the small  $s$  expansion of  $K(x; s)$ . In  $D$  dimensions  $K(x; s)$  has a small  $s$  expansion of the form (see *e.g.* [51])

$$K(x; s) = \sum_{n=0}^{\infty} K_{-\frac{D}{2}+n}(x) s^{-\frac{D}{2}+n}, \quad (2.20)$$

where the coefficients  $K_{-\frac{D}{2}+n}(x)$  are given by local general coordinate and gauge invariant combinations of the background fields containing  $2n$  derivatives, *e.g.*  $K_{-D/2}(x)$  is a constant,  $K_{-D/2+1}$  is a linear combinations of  $R$ ,  $F_{\mu\nu}^{(\alpha)} F^{(\alpha')\mu\nu}$ ,  $\partial_{\mu} \varphi_s \partial^{\mu} \varphi_{s'}$  etc. The logarithmic correction comes from the order  $s^0$  terms in this expansion. Using (2.19) and (2.20) we get

$$\ln Z_{\text{nz}} = \ln a (C_{\text{local}} - N_{\text{zm}}) + \dots, \quad (2.21)$$

where

$$C_{\text{local}} = \int d^D x \sqrt{\det g} K_0(x), \quad (2.22)$$

and  $\dots$  denotes other terms which do not have  $\ln a$  factors but are possibly divergent in the  $\epsilon \rightarrow 0$  limit. The significance of these divergent terms will be explained shortly. (2.20) shows that  $K_0(x)$  and hence  $C_{\text{local}}$  vanishes in odd dimensions. In even dimensions  $K_0(x)$  contains  $D$  derivatives and as a result  $C_{\text{local}}$  is a function of the variables  $(\beta, \vec{\omega}, \vec{\mu})$  which scales as  $\Lambda^0$  under the scaling (2.2).

Before we proceed, a few comments are in order.

1. Special mention must be made of the terms in (2.20) which diverge in the  $\epsilon \rightarrow 0$  limit. The interpretation of these divergences was discussed in [2]. A term of the form  $K_{-\alpha}(x)s^{-\alpha}$  gives a contribution

$$\frac{1}{2} \int d^D x \sqrt{\det g} \int_{\epsilon}^{\epsilon a^2/a_0^2} \frac{ds}{s} K_{-\alpha}(x) s^{-\alpha} = \frac{1}{2} \int d^D x \sqrt{\det g} K_{-\alpha}(x) \frac{1}{\alpha} \epsilon^{-\alpha} (1 - (a/a_0)^{-2\alpha}). \quad (2.23)$$

Now since  $K_{-\alpha}$  contains  $D/2 - \alpha$  derivatives, it follows from the scaling symmetry of the theory that  $\int d^D x \sqrt{\det g} K_{-\alpha}(x)$  has a dependence of the form  $a^{2\alpha}$ . Thus (2.23) goes as  $a^{2\alpha}/\epsilon^\alpha - a_0^{2\alpha}/\epsilon^\alpha$  and is divergent for  $\alpha > 0$  if we insist on taking the  $\epsilon \rightarrow 0$  limit instead of keeping  $\epsilon$  of the order of Planck length square. What is the origin of these divergences? Clearly the  $a^{2\alpha}$  term comes from the original black hole solution and the  $a_0^{2\alpha}$  term comes from the black hole of size  $a_0$ ; so it is sufficient to focus on the  $a^{2\alpha}$  terms. The divergent coefficient of this term in the  $\epsilon \rightarrow 0$  limit can be traced to the usual ultraviolet divergences in field theory which renormalize various parameters of the theory. For example the leading divergent term, corresponding to  $\alpha = D/2$ , has  $K_{-\alpha}(x)$  an  $x$  independent constant. The corresponding divergent contribution to (2.23) can be written as

$$\frac{1}{2} \frac{K_{-D/2}}{D/2} \epsilon^{-D/2} \int d^D x \sqrt{\det g}. \quad (2.24)$$

(2.24) clearly has the interpretation of a one loop contribution to the Euclidean effective action of the form  $-\frac{1}{2} \frac{K_{-D/2}}{D/2} \epsilon^{-D/2} \int d^D x \sqrt{\det g}$  – a cosmological constant term. Since the theory we consider by assumption does not have a cosmological constant this must be removed by a counterterm. The same counterterm will also remove the corresponding divergent contribution to (2.19). Similarly the first subleading divergent contributions proportional to  $\epsilon^{-D/2+1} K_{-D/2+1}(x)$  can be interpreted as the result of renormalization of the coefficients of various two derivative terms in the action, *e.g.*  $R$ ,  $F_{\mu\nu}^{(\alpha)} F^{(\alpha')\mu\nu}$ ,  $\partial_\mu \varphi_s \partial^\mu \varphi_{s'}$  etc. Again these divergences must be removed by adding counterterms to the

action, and these will have the effect of removing the corresponding divergences from (2.19).<sup>6</sup> This way all the divergent contributions to  $\ln Z$  are removed by adding to the action the same local counterterms which are needed to get finite results for physical quantities independently of the computation of black hole entropy. Alternatively if the theory comes with an intrinsic ultraviolet cut-off that makes  $\epsilon$  of the order of Planck length square, then these contributions can be absorbed into a finite renormalization of the various coupling constants of the theory.

2. Since we have put the black hole inside a box of size  $L$  with appropriate boundary conditions on various fields at the boundary of the box, the small  $s$  expansion of  $K(x, s)$  also contains terms which are localized on the boundary instead of the bulk of space-time [51]. These contributions to the partition function do not have anything to do with the black hole, and arise from boundary effects. Thus they must be removed from  $\ln Z$ . Formally this will be done by explicitly removing from  $K(x; s)$  the boundary terms carrying non-positive powers of  $s$  in the small  $s$  expansion. This significance of this subtraction will be explained in appendix B. The positive powers of  $s$  on the other hand will give negligible contribution to (2.19).
3. Eq.(2.19), (2.20) shows that in general the one loop correction to  $\ln Z$  and hence the entropy depends on the form of the black hole solution over the entire space-time, and not just the near horizon geometry. This would seem to be in apparent conflict with computations based on Wald's formula applied to the quantum effective action or entanglement entropy computation reviewed in [27], which depend only on the near horizon geometry. We must however keep in mind that the near horizon geometry of the black hole, expressed as a function of the asymptotic parameters like temperature, chemical potential etc., can get corrected due to quantum corrections to the effective action and these corrections are controlled by the form of the original solution over the entire space-time. In particular the corrections to the near horizon geometry could involve terms proportional to  $\ln a$  and hence the usual Bekenstein-Hawking entropy evaluated in the new background could have additional logarithmic corrections. Thus in order to compute the logarithmic correction to the entropy we must use information about the full black

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<sup>6</sup>If the theory contains equal number of bosonic and fermionic degrees of freedom, *e.g.* a supersymmetric theory, then there is no one loop contribution to the renormalization of the cosmological constant and  $K_{-D/2}$  vanishes. In some extended supergravity theories the one loop correction to two derivative terms also vanish. In this case  $K_{-D/2+1}$  is also zero.

hole solution in all the approaches. Extremal black holes are exceptional since for them the attractor mechanism allows us to fix the near horizon geometry without knowing the details of the full solution. Also for uncharged black holes in  $D = 4$  the analysis simplifies since  $K_0(x)$  is just the Euler density and hence its integral is determined by the topology of the solution. At the same time it does not affect the equations of motion and hence does not introduce any correction proportional to  $\ln a$  to the near horizon field configurations.

4. It is instructive to identify the region of loop momentum integration that is responsible for the  $\ln a$  terms in the partition function. For this we shall take the ultraviolet cut-off  $\epsilon$  to be of order unity, i.e. of the order of the square of the Planck length. Then the  $\ln a$  contribution to (2.19) comes from the range  $1 \ll s \ll a^2/a_0^2$ . In terms of loop momentum – which is of order  $1/\sqrt{s}$  – this means that the logarithmic corrections come from the range of loop momentum integration which is much less than the Planck mass. Hence it involves infrared physics.

Let us now briefly discuss the evaluation of  $K_0(x)$ . It follows from (2.20) that  $K_0(x)$  vanishes for odd  $D$ . In even dimensions we can evaluate  $K_0(x)$  in a general black hole background using the general method described in [51], which allows us to express  $K_0(x)$  as a linear combinations of covariant terms, each containing  $D$  derivatives. Thus for example in  $D = 4$ ,  $K_0(x)$  will have the form

$$K_0(x) = \alpha R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + \dots, \quad (2.25)$$

where  $\alpha, \beta, \gamma$  etc. are computable coefficients. For Einstein-Maxwell theory in four space-time dimensions this has been calculated recently in [26] with the result:

$$K_0(x) = \frac{1}{360 \times 16\pi^2} (398 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 52 R_{\mu\nu} R^{\mu\nu}) . \quad (2.26)$$

If in addition we have  $n_S$  scalars,  $n_F$  Dirac fermions,  $(n_V - 1)$  more vector fields and  $n_{3/2}$  spin 3/2 fields, all minimally coupled to background gravity and no coupling to the background gauge field, then (2.26) is modified to [44–52]

$$K_0(x) = \frac{1}{360 \times 16\pi^2} \left\{ \left( 398 + 2n_S - 26(n_V - 1) + 7n_F - \frac{233}{2}n_{3/2} \right) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + (52 - 2n_S + 176(n_V - 1) + 8n_F + 233n_{3/2}) R_{\mu\nu} R^{\mu\nu} \right\}. \quad (2.27)$$



As an illustration we shall now compute  $K_0(x)$  for a Kerr-Newmann black hole. The metric of a general Kerr-Newmann black hole of mass  $M$ , angular momentum  $J$  and charge  $Q$  (in appropriate units) is given by

$$\begin{aligned}
ds^2 &= -\frac{r^2 + b^2 \cos^2 \psi - 2Mr + Q^2}{r^2 + b^2 \cos^2 \psi} dt^2 + \frac{r^2 + b^2 \cos^2 \psi}{r^2 + b^2 - 2Mr + Q^2} dr^2 + (r^2 + b^2 \cos^2 \psi) d\psi^2 \\
&\quad + \frac{(r^2 + b^2 \cos^2 \psi)(r^2 + b^2) + (2Mr - Q^2)b^2 \sin^2 \psi}{r^2 + b^2 \cos^2 \psi} \sin^2 \psi d\phi^2 \\
&\quad + \frac{2(Q^2 - 2Mr)b}{r^2 + b^2 \cos^2 \psi} \sin^2 \psi dt d\phi \\
b &= \frac{J}{M}.
\end{aligned} \tag{2.28}$$

The location  $r_H$  of the horizon and the classical Bekenstein-Hawking entropy  $S_{BH}$  are given respectively by

$$r_H = M + \sqrt{M^2 - Q^2 - b^2} = \frac{1}{M}(M^2 + \sqrt{M^4 - Q^2 M^2 - J^2}), \tag{2.29}$$

and

$$S_{BH} = \pi(2M^2 - Q^2 + 2M\sqrt{M^2 - (b^2 + Q^2)}) = \pi(2M^2 - Q^2 + 2\sqrt{M^4 - (J^2 + Q^2 M^2)}). \tag{2.30}$$

Using (2.4) we now get

$$\begin{aligned}
\beta &= \frac{2\pi M}{\sqrt{M^4 - J^2 - M^2 Q^2}} \left\{ 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - M^2 Q^2} \right\}, \\
\omega &= -\frac{2\pi J}{\sqrt{M^4 - J^2 - M^2 Q^2}}, \\
\mu &= -\frac{2\pi Q}{\sqrt{M^4 - J^2 - M^2 Q^2}} \left\{ M^2 + \sqrt{M^4 - J^2 - M^2 Q^2} \right\}.
\end{aligned} \tag{2.31}$$

Now for the background (2.28) we have [54, 55]

$$\begin{aligned}
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \frac{8}{(r^2 + b^2 \cos^2 \psi)^6} \left\{ 6M^2(r^6 - 15b^2 r^4 \cos^2 \psi + 15b^4 r^2 \cos^4 \psi - b^6 \cos^6 \psi) \right. \\
&\quad \left. - 12MQ^2 r(r^4 - 10r^2 b^2 \cos^2 \psi + 5b^4 \cos^4 \psi) \right. \\
&\quad \left. + Q^4(7r^4 - 34r^2 b^2 \cos^2 \psi + 7b^4 \cos^4 \psi) \right\} \\
R_{\mu\nu} R^{\mu\nu} &= \frac{4Q^4}{(r^2 + b^2 \cos^2 \psi)^4}, \\
\det g &= (r^2 + b^2 \cos^2 \psi)^2 \sin^2 \psi.
\end{aligned} \tag{2.32}$$

After analytic continuation  $t \rightarrow -i\tau$  and identifying  $\tau$  as a periodic variable with period  $\beta$  we get,

$$\begin{aligned} & \int d^4x \sqrt{\det g} R_{\mu\nu} R^{\mu\nu} \\ &= \frac{\pi\beta Q^4}{2b^5 r_H^4 (b^2 + r_H^2)} \left\{ 3b^5 r_H + 2b^3 r_H^3 + 3(b^2 - r_H^2)(b^2 + r_H^2)^2 \tan^{-1}\left(\frac{b}{r_H}\right) + 3br_H^5 \right\} \\ & \int d^4x \sqrt{\det g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 64\pi^2 + 4 \int d^4x \sqrt{\det g} R_{\mu\nu} R^{\mu\nu} \end{aligned} \quad (2.33)$$

Substituting these into eqs.(2.22), (2.27) we get

$$\begin{aligned} C_{\text{local}} &= \frac{1}{90} \left( 2n_S - 26(n_V - 1) + 7n_F - \frac{233}{2} n_{3/2} + 398 \right) \\ &+ \frac{(1644 + 6n_S + 72(n_V - 1) + 36n_F - 233n_{3/2})}{360 \times 16\pi^2} \frac{\pi\beta Q^4}{2b^5 r_H^4 (b^2 + r_H^2)} \\ &\times \left\{ 3b^5 r_H + 2b^3 r_H^3 + 3(b^2 - r_H^2)(b^2 + r_H^2)^2 \tan^{-1}\left(\frac{b}{r_H}\right) + 3br_H^5 \right\}. \end{aligned} \quad (2.34)$$

We can now consider some special cases of this formula:

1. For an uncharged black hole we have  $Q = 0$  and (2.34) reduces to

$$C_{\text{local}} = \frac{1}{90} \left( 2n_S - 26(n_V - 1) + 7n_F - \frac{233}{2} n_{3/2} + 398 \right). \quad (2.35)$$

Since this is valid for all angular momentum, it is also valid for Schwarzschild black holes. This agrees with the standard results in the literature, see *e.g.* [3].

2. For charged non-rotating black hole we have  $b = 0$ . Taking the  $b \rightarrow 0$  limit of (2.34) we get

$$\begin{aligned} C_{\text{local}} &= \frac{1}{90} \left( 2n_S - 26(n_V - 1) + 7n_F - \frac{233}{2} n_{3/2} + 398 \right) \\ &+ \frac{\beta Q^4}{1800\pi r_H^5} (1644 + 6n_S + 72(n_V - 1) + 36n_F - 233n_{3/2}). \end{aligned} \quad (2.36)$$

## 2.4 One loop contribution to the partition function from the zero modes

Next we turn to the evaluation of the number  $N_{\text{zm}}$  of zero modes and their contribution  $Z_{\text{zm}}$  to the partition function. The zero modes of the Lorentzian black hole solution are associated with

the translation and rotation symmetries which are broken by the black hole. It has been argued in appendix A that only those translations which are invariant under the rotation generator  $\vec{\omega} \cdot \vec{T}$  generate zero modes of the Euclidean black hole solution, – the other translational zero modes and all the rotational zero modes of the Lorentzian solution fail to satisfy the required periodicity along the Euclidean time direction and hence are lifted. We shall denote by  $n_T$  the number of translational zero modes of the Euclidean solution. Thus for example in four space-time dimensions  $n_T = 3$  for non-rotating black holes since for these  $\vec{\omega} \cdot \vec{T} = 0$  and hence all three broken translation symmetries generate zero modes. But for a black hole rotating along the  $z$ -axis we have  $\vec{\omega} \cdot \vec{T} = T_3$  and hence only the translation along the third direction generates a zero mode. Thus in this case we have  $n_T = 1$ . A rotating black hole also breaks part of the rotational invariance, but as mentioned above there are no zero modes of the Euclidean black hole solution associated with the broken rotational invariance.

Let  $h_{\mu\nu}$  denote the fluctuating gravitons in the black hole background. We normalize the path integral measure as

$$\int [Dh_{\mu\nu}] \exp \left[ - \int d^D x \sqrt{\det g} g^{\mu\nu} g^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma} \right] = 1. \quad (2.37)$$

Now in an appropriate coordinate system the metric  $g_{\mu\nu}$  has the form  $a^2 g_{\mu\nu}^{(0)}$  where  $g_{\mu\nu}^{(0)}$  is  $a$  independent. Then we can express (2.37) as

$$\int [Dh_{\mu\nu}] \exp \left[ -a^{D-4} \int d^D x \sqrt{\det g^{(0)}} g^{(0)\mu\nu} g^{(0)\rho\sigma} h_{\mu\rho} h_{\nu\sigma} \right] = 1. \quad (2.38)$$

Thus the correctly normalized integration measure, up to an  $a$  independent constant, is  $\prod_{x,(\mu\nu)} d(a^{(D-4)/2} h_{\mu\nu}(x))$ . Now the translational zero modes are associated with diffeomorphisms with non-normalizable parameters  $\xi^{(i)\mu}$  such that  $\xi^{(i)\mu} \rightarrow \delta_{i\mu}$  asymptotically but vanishes below a certain radius. We can introduce parameters  $u_{(i)}$  labelling the zero mode deformations via

$$h_{\mu\nu} = u_{(i)} (D_\mu \xi_\nu^{(i)} + D_\nu \xi_\mu^{(i)}) . \quad (2.39)$$

Our strategy will be to first transform integration over the metric variables to integration over  $u_{(i)}$  and then find the  $a$  dependence of the range of integration over the  $u_{(i)}$ 's.  $D_\mu \xi_\nu^{(i)} + D_\nu \xi_\mu^{(i)}$  is the Jacobian of change of variables from  $h_{\mu\nu}$  to  $u_{(i)}$ . Although with our choice of normalization  $\xi^{(i)\mu}$  is  $a$  independent, lowering the index makes  $\xi_\mu^{(i)} \sim a^2$ . Thus for each zero mode the Jacobian of change of variables from  $a^{(D-4)/2} h_{\mu\nu}$  to  $u_{(i)}$  gives a factor of  $a^{\frac{D-4}{2}+2} = a^{D/2}$ . Next we need to find the  $a$  dependence of the integration range over the  $u_{(i)}$ 's. Since  $\xi^{(i)\mu} \rightarrow \delta_{i\mu}$  asymptotically,

the integration range of  $u_{(i)}$  corresponds to the range of the coordinate  $x^i$  in which we confine the black hole. Let  $L$  be the proper size of the box in which we confine the black hole – for simplicity we shall take this to be the same in all directions. Now in the coordinate system in which the metric has the form  $a^2 g_{\mu\nu}^{(0)}$ , the asymptotic metric is  $a^2 \eta_{\mu\nu}$ . Thus the range of the coordinate  $x^i$  is given by  $L/a$ , and hence the integration range over  $u_{(i)}$  is also given by  $L/a$ . Combining this with the Jacobian factor  $a^{D/2}$  found earlier we see that integration over each  $u_{(i)}$  produces a factor of  $a^{D/2}(L/a)$ . Thus the net contribution from  $n_T$  such zero modes to  $\ln Z$  is given by

$$\ln Z_{\text{zm}} = \frac{D}{2} n_T \ln a + n_T \ln \frac{L}{a}. \quad (2.40)$$

Using (2.21), (2.40) and  $N_{\text{zm}} = n_T$  we can express (2.12) as

$$\ln Z = \ln Z_{\text{cl}} + n_T \ln L + \left( C_{\text{local}} + \frac{1}{2} n_T (D - 4) \right) \ln a + \dots. \quad (2.41)$$

## 2.5 Higher loop contributions

Since we have only analyzed the contribution from one loop determinants of massless fields it is appropriate to ask if higher loop corrections could change the result. In  $D$  dimensions naive power counting shows that the  $\ell$ -loop vacuum graph has a net mass dimension of  $(D - 2)\ell + 2$ , so that after multiplying this by  $(\ell - 1)$  powers of Newton's constant  $l_p^{D-2}$  we get a term of mass dimension  $D$  – the required dimension of a Lagrangian density. The contribution to  $\ln Z$  is obtained by multiplying this by a factor of  $\sqrt{\det g} \sim a^D$  and then integrating this over the black hole space-time. Furthermore in the presence of the black hole background the various propagators and vertices carrying momenta  $k$  are modified from their form in flat space-time background by multiplicative functions of  $ka$  which approach 1 for large values of  $ka$ , so that for large momentum we recover the propagators and vertices in flat space-time background. Putting these results together we see that the  $\ell$ -loop contribution to  $\ln Z$  may be schematically written as

$$l_p^{(D-2)(\ell-1)} a^D \int^{1/\sqrt{\epsilon}} d^{D\ell} k k^{2-2\ell} F(ka), \quad (2.42)$$

where  $F(ka)$  is some function that approaches 1 for large value of its argument, and the factor of  $a^D$  outside the integral comes from the  $\sqrt{\det g}$  factor. The power of  $k$  inside the integral has been adjusted so that  $d^{D\ell} k k^{2-2\ell}$  has mass dimension  $(D - 2)\ell + 2$ . The upper limit  $1/\sqrt{\epsilon}$  on the integral indicates that the ultraviolet cut-off on the loop momentum integral is taken

to be of order  $1/\sqrt{\epsilon}$ . Using a change of variables  $\tilde{k} = ka$  we can express (2.42) as

$$l_p^{(D-2)(\ell-1)} a^{-(D-2)(\ell-1)} \int^{a/\sqrt{\epsilon}} d^D \tilde{k} \tilde{k}^{2-2\ell} F(\tilde{k}). \quad (2.43)$$

First consider the case where all loop momenta are of the same order. Since  $F(\tilde{k}) \rightarrow 1$  for large  $\tilde{k}$ , we can expand it in a power series in  $\tilde{k}^{-1}$ . Possible  $\ln a$  term will come from the order  $\tilde{k}^{-(D-2)\ell-2}$  term in this expansion. But the corresponding  $\ln(a/\sqrt{\epsilon})$  term is multiplied by  $a^{-(D-2)(\ell-1)}$  and hence is suppressed for large  $a$  except for  $\ell = 1$ . This shows that we do not get any logarithmic correction to the entropy from the region of loop momentum integration where all momenta are of the same order. Next consider the possibility where a subset of the loop momenta are smaller than the rest; we shall call the part of the graph that carries low momentum soft part and the rest hard part. In that case we can regard the effect of the hard loops as renormalization of the vertices and propagators of the soft part of the graph, and the contribution from such graphs essentially reduces to a lower order contribution where soft lines appear as propagators and all hard lines are collapsed into renormalization of vertices and propagators. Thus as long as this renormalization does not change the low energy effective action *e.g.* the massless particles are kept massless and minimal coupling to gravity remains minimal (by adding explicit counterterms if necessary) these contributions do not change logarithmic corrections to the black hole entropy. This argument holds in particular in a theory of pure gravity, since gravitons must remain massless even after quantum corrections if they are to describe the long range gravitational force that we see in nature. The renormalization effects can also generate higher derivative couplings, but higher derivative corrections will give additional powers of  $l_p/a$ , making the coefficient of the  $\ln(a/\sqrt{\epsilon})$  term suppressed by even more powers of  $l_p/a$  than what has been argued before. Thus we conclude that as long as the massless fields are kept massless and minimally coupled to gravity even after renormalization effects are taken into account, the one loop logarithmic correction to the partition function is not altered by higher loop corrections.

## 2.6 Effect of cosmological constant term

Finally let us discuss briefly how the analysis changes in the presence of a cosmological constant term. In this case the eigenvalues of the kinetic operator will be of order  $l^{-2}$  where  $l$  is some length scale set by the cosmological constant, and the presence of a black hole of size  $a$  introduces small corrections of order  $a^{-2}$  to these eigenvalues. Thus the integration over the proper time

variable  $s$  will be suppressed exponentially for  $s \gg l^2$  and there is no contribution of order  $\ln a$  from the region  $l_p^2 \ll s \ll a^2$ . As a result there is no logarithmic correction to  $\ln Z$  from the non-zero modes. Depending on the situations there may be some logarithmic correction from the zero modes, but often, as in the case of BTZ black holes to be discussed in §5, even the zero mode contributions are absent. In such cases there is no correction of order  $\ln a$  to  $\ln Z$ . However as we shall see in §3 the entropy can still receive logarithmic corrections during the process of converting the partition function into entropy.<sup>7</sup>

### 3 Black hole entropy in microcanonical and other ensembles

The  $Z$  computed in §2 describes the partition function computed from the gravity side. This should be identified as the statistical grand canonical partition function  $Z_{stat}$

$$Z_{stat} = Tr \left( e^{-\beta E - \vec{\omega} \cdot \vec{J} - \vec{\mu} \cdot \vec{Q}} \right), \quad (3.1)$$

where the trace runs over all the black hole microstates carrying different mass, charges, angular momenta and momenta  $\vec{P}$ .  $E = M + \vec{P}^2/2M$  denotes the total energy of the black hole.  $Z_{stat}$  can be computed from microstate degeneracies whenever the latter results are available, which can then be compared with the partition function  $Z$  computed from the gravity side. Alternatively, by equating  $Z_{stat}$  with the Euclidean gravity prediction (2.41) for  $Z$  we can arrive at definite predictions for the black hole entropy in the microcanonical (or, in any other) ensemble. This can then be compared with the microscopic results for the same quantities if and when the microscopic results are available. We shall follow the latter point of view and, from now on, identify  $Z_{stat}$  with  $Z$ . Our goal in this section will be to convert the result for  $Z$  obtained in §2 to entropies defined in different ensembles which may be relevant for comparison with the microscopic results.

#### 3.1 Entropy in the microcanonical ensemble

Before we proceed we need to clarify the meaning of the entropy in microcanonical ensemble. Since the charges are quantized in integer units, it is possible to fix their values at definite numbers and count states to define the entropy. For angular momentum we cannot fix all the

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<sup>7</sup>The effect of the change in the ensemble on the logarithmic corrections to the entropy has been analyzed before, see *e.g.* [43, 56–59].

components, but we can fix the components associated with the Cartan generators to specific integer or half integer values. This cannot be done for the mass since the quantization rules for the mass is not known without knowing the microscopic details. For this reason we define the microcanonical entropy such that

$$e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})} \delta M \quad (3.2)$$

represents the number of internal microstates of the black hole in the mass range  $\delta M$ , carrying charge  $\vec{Q}$ , angular momentum  $\vec{J}$  and vanishing total momentum.<sup>8</sup> The interval  $\delta M$  appearing in (3.2) must be small enough so that  $S_{\text{mc}}$  does not vary appreciably over the interval, and yet large enough so as to contain a large number of states and so as to be larger than the decay width of individual microstates. This can be achieved by taking the size of the black hole to be sufficiently large.

Since  $S_{\text{mc}}$  does not depend on the momentum  $\vec{P}$  as a consequence of Lorentz invariance, we can perform the sum over  $\vec{P}$  implicit in (3.1) explicitly. If  $\vec{P}$  is not invariant under  $\vec{\omega} \cdot \vec{J}$ , then  $e^{-\vec{\omega} \cdot \vec{J}}$  acting on the state will produce a state with a different momentum<sup>9</sup> and hence the contribution of this state to the trace will vanish. For this reason we must restrict the momentum of the state to be along the directions invariant under  $\vec{\omega} \cdot \vec{J}$ . The number of such momentum components is the number  $n_T$  introduced in §2 (see the discussion in the paragraph above (2.37)). If we assume as in §2 that the black hole is confined in a box of length  $L$  along each of these  $n_T$  directions then the trace over the momentum along these directions can be represented, up to a numerical factor, by  $L^{n_T} \int d^{n_T} P$ . With this, the statistical grand canonical partition function  $Z$  is related to  $S_{\text{mc}}$  via the relation

$$\begin{aligned} Z(\beta, \vec{\omega}, \vec{\mu}) &\sim L^{n_T} \int dM d^{n_T} P \sum_{\vec{J}, \vec{Q}} e^{-\beta M - \beta(\vec{P}^2/2M) - \vec{\omega} \cdot \vec{J} - \vec{\mu} \cdot \vec{Q}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})} \\ &= L^{n_T} \int dM \left( \frac{2\pi M}{\beta} \right)^{n_T/2} \sum_{\vec{J}, \vec{Q}} e^{-\beta M - \vec{\omega} \cdot \vec{J} - \vec{\mu} \cdot \vec{Q}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})}. \end{aligned} \quad (3.3)$$

Clearly the integrand / summand in the right hand side of (3.3) is sharply peaked around the classical values of  $M, \vec{J}, \vec{Q}$  given by the solutions to (2.4). Since  $(2\pi M/\beta)^{n_T/2}$  is a smooth

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<sup>8</sup>In contrast the entropy of extremal black holes analyzed in [18–22, 26] correspond to ground state degeneracy in a given charge sector, and not the number of states in a given range of mass. Thus we cannot directly compare our results with those of [18–22, 26]. In any case since the present analysis has been carried out under the assumption that all length scales are of the same order – in particular  $\beta \sim a$  – the results are not necessarily valid in the extremal limit in which  $\beta \rightarrow \infty$  keeping  $a$  fixed.

<sup>9</sup>For this we analytically continue  $\omega_a$  to imaginary values so that  $e^{-\vec{\omega} \cdot \vec{J}}$  represents a rotation group element.

functions of  $M, \vec{J}, \vec{Q}$ , we can replace it by its classical value and take it out of the integral. Eqs.(2.2), (2.5) now shows that it scales as  $a^{(D-4)n_T/2}$ . Substituting this into (3.3) and using (2.41) we get

$$\begin{aligned} & \exp \left[ \ln Z_{\text{cl}}(\beta, \vec{\omega}, \vec{\mu}) + n_T \ln L + \left( C_{\text{local}} + \frac{1}{2} n_T (D-4) \right) \ln a \right] \\ & \sim L^{n_T} a^{(D-4)n_T/2} \int dM \sum_{\vec{J}, \vec{Q}} e^{-\beta M - \vec{\omega} \cdot \vec{J} - \vec{\mu} \cdot \vec{Q}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})}, \end{aligned} \quad (3.4)$$

i.e.

$$\exp [\ln Z_{\text{cl}}(\beta, \vec{\omega}, \vec{\mu}) + C_{\text{local}} \ln a] \sim \int dM \sum_{\vec{J}, \vec{Q}} e^{-\beta M - \vec{\omega} \cdot \vec{J} - \vec{\mu} \cdot \vec{Q}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})}. \quad (3.5)$$

Note that the explicit dependence on  $n_T$  has cancelled so that the left hand side (and hence also the right hand side) of this equation has no discontinuity at special values of  $\vec{\omega}$ , *e.g.*  $\vec{\omega} = 0$ , where there is enhanced rotational symmetry and consequently an increase in the value of  $n_T$ . We can formally invert (3.5) to write:

$$e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})} \sim \int d\beta d^{N_C} \omega d^{n_V} \mu \exp \left[ \beta M + \vec{\omega} \cdot \vec{J} + \vec{\mu} \cdot \vec{Q} + \ln Z_{\text{cl}}(\beta, \vec{\omega}, \vec{\mu}) + C_{\text{local}} \ln a \right], \quad (3.6)$$

where  $N_C$  is the total number of Cartan generators, i.e. the rank of the rotation group and  $n_V$  is the total number of Maxwell fields. (3.5) and (3.6) are equivalent as long as it is understood that all the integrals and sums are evaluated using saddle point approximation. To leading order the location of the saddle point of (3.6) is at  $(\beta, \vec{\omega}, \vec{\mu})$  satisfying

$$M = -\frac{\partial \ln Z_{\text{cl}}}{\partial \beta}, \quad J_b = -\frac{\partial \ln Z_{\text{cl}}}{\partial \omega_b}, \quad Q_\alpha = -\frac{\partial \ln Z_{\text{cl}}}{\partial \mu_\alpha}. \quad (3.7)$$

These relations are the same as those given in (2.4). The leading contribution to  $S_{\text{mc}}$  is given by  $\ln Z_{\text{cl}} + \beta M + \vec{\omega} \cdot \vec{J} + \vec{\mu} \cdot \vec{Q}$  evaluated at the saddle point, which according to (2.4) is the same as  $S_{\text{BH}}$ .

In order to study the effect of one loop correction  $C_{\text{local}} \ln a$  on  $S_{\text{BH}}$  we can ignore the effect of these corrections on the saddle point values of  $\beta$ ,  $\vec{\omega}$  and  $\vec{\mu}$ , since at the leading order the integrand has been extremized with respect to these variables at the saddle point, and hence the effect of any change in the location of the saddle point will affect the result at the second order. Thus the correction to  $S_{\text{mc}}$  comes from two sources:

- $C_{\text{local}} \ln a$  evaluated at the saddle point, and



- the Gaussian integral over  $\beta, \vec{\omega}, \vec{\mu}$  around the saddle point.

To compute the result of Gaussian integration we note that from (2.5) we get

$$\frac{\partial^2 \ln Z_{\text{cl}}}{\partial \beta^2} \sim a^{D-4}, \quad \frac{\partial^2 \ln Z_{\text{cl}}}{\partial \omega_b \partial \omega_c} \sim a^{D-2}, \quad \frac{\partial^2 \ln Z_{\text{cl}}}{\partial \mu_\alpha \partial \mu_\beta} \sim a^{D-4}, \quad (3.8)$$

and the mixed derivatives scale accordingly. If the second derivative matrix does not have accidental zero eigenvalues, then (3.8) allows us to express  $Z$  as a gaussian peaked around the saddle point, with the following widths for the different variables:<sup>10</sup>

$$\Delta \beta \sim a^{-(D-4)/2}, \quad (3.9)$$

$$\Delta \omega_b \sim a^{-(D-2)/2}, \quad (3.10)$$

$$\Delta \mu_\alpha \sim a^{-(D-4)/2}. \quad (3.11)$$

Thus integration over each of these variables will give a contribution to the right hand side of (3.6) of this order. Combining these results we get

$$S_{\text{mc}} = S_{\text{BH}} + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-2)N_C - \frac{1}{2}(D-4)n_V \right). \quad (3.12)$$

Note that this final formula for  $S_{\text{mc}}$  is independent of  $n_T$ . This can be traced to the cancelation of explicit  $n_T$  dependent factors on the two sides of (3.4). The absence of  $n_T$  dependence in (3.12) shows that while the grand canonical partition function has a discontinuity at  $\vec{\omega} = 0$  where the black hole has enhanced rotational symmetry, there is no need for such a discontinuity in  $S_{\text{mc}}$  at  $\vec{J} = 0$ . The discontinuity in the gravitational partition function  $Z$  due to the increased number of zero modes at  $\vec{\omega} = 0$  has its counterpart in the statistical partition function due to the explicit  $n_T$  dependent factors on the right hand side of (3.3) rather than in any discontinuity in the function  $S_{\text{mc}}$ . In particular this indicates that the asymptotic expansion of  $S_{\text{mc}}(M, \vec{J}, \vec{Q})$  for large charges can be regarded as an analytic function of  $\vec{J}$  even around  $\vec{J} = 0$ .

## 3.2 Entropy in other ensembles

For spherically symmetric black holes it is often convenient to work in a mixed ensemble where we keep the charges and mass fixed but sum over all possible angular momentum states. If we

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<sup>10</sup>This requires the matrix of second derivatives to be negative definite, i.e. the saddle point to be a maximum. However since Schwarzschild black hole has negative specific heat this is not quite true. Nevertheless we shall proceed by assuming that the integral can be evaluated using saddle point approximation by appropriately rotating the integration contours into the complex plane.

denote by  $S_{\text{mixed}}(M, \vec{Q})$  the corresponding entropy then we have

$$e^{S_{\text{mixed}}(M, \vec{Q})} = \sum_{\vec{J}} e^{S_{\text{mc}}(M, \vec{J}, \vec{Q})} . \quad (3.13)$$

The analog of eqs.(3.5), (3.6) now take the forms:

$$\exp [\ln Z_{\text{cl}}(\beta, \vec{\omega} = 0, \vec{\mu}) + C_{\text{local}} \ln a] \sim \int dM \sum_{\vec{Q}} e^{-\beta M - \vec{\mu} \cdot \vec{Q}} e^{S_{\text{mixed}}(M, \vec{Q})} , \quad (3.14)$$

$$e^{S_{\text{mixed}}(M, \vec{Q})} \sim \int d\beta d^{n_V} \mu \exp \left[ \beta M + \vec{\mu} \cdot \vec{Q} + \ln Z_{\text{cl}}(\beta, \vec{\omega} = 0, \vec{\mu}) + C_{\text{local}} \ln a \right] . \quad (3.15)$$

Evaluating the integral using saddle point approximation as before we get

$$S_{\text{mixed}}(M, \vec{Q}) = S_{\text{BH}}(M, \vec{J} = 0, \vec{Q}) + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-4)n_V \right) . \quad (3.16)$$

We could also consider the other extreme in which we require all components of the angular momentum – not just those associated with the Cartan generators – to vanish. This is equivalent to requiring that we count only singlet states of the rotation group. The corresponding entropy, denoted as  $S_{\text{singlet}}(M, \vec{Q})$  can be calculated as follows. Let us denote by  $N_R = (D-1)(D-2)/2$  the dimension of the rotation group, by  $\{\chi^1, \dots, \chi^{N_R}\}$  the parameters labeling a rotation group element (with  $\vec{\chi} = 0$  being the identity element) and by  $d^{N_R} \chi$  the Haar measure of the group. Furthermore let us denote by  $e^{i\vec{\theta} \cdot \vec{T}}$  the element of the Cartan subgroup conjugate to  $\vec{\chi}$  with  $T^a$ 's being the Cartan generator. Then since  $\exp[S_{\text{mc}}(M, \vec{J}, \vec{Q})]$  is the number of states with eigenvalues  $\vec{J}$  under the Cartan generators, the character of the representation of the rotation group formed by the black hole microstates will be given by

$$\sum_{\vec{J}} \exp[S_{\text{mc}}(M, \vec{J}, \vec{Q}) + i\vec{\theta} \cdot \vec{J}] . \quad (3.17)$$

From this we can extract the number of singlet states as

$$\exp[S_{\text{singlet}}(M, \vec{Q})] = \int d^{N_R} \chi \sum_{\vec{J}} \exp[S_{\text{mc}}(M, \vec{J}, \vec{Q}) + i\vec{\theta} \cdot \vec{J}] , \quad (3.18)$$

where it is understood that  $\theta^a$ 's are functions of  $\vec{\chi}$ . Now using (3.5) we get

$$\sum_{\vec{J}} \exp[S_{\text{mc}}(M, \vec{J}, \vec{Q}) + i\vec{\theta} \cdot \vec{J}] \sim \int d\beta d^{n_V} \mu \exp \left[ \beta M + \vec{\mu} \cdot \vec{Q} + \ln Z_{\text{cl}}(\beta, \vec{\omega} = i\vec{\theta}, \vec{\mu}) + C_{\text{local}} \ln a \right] . \quad (3.19)$$

Using this we can express (3.18) as

$$\exp[S_{\text{singlet}}(M, \vec{Q})] \sim \int d\beta d^{N_R} \chi d^{n_V} \mu \exp \left[ \beta M + \vec{\mu} \cdot \vec{Q} + \ln Z_{\text{cl}}(\beta, \vec{\omega} = i\vec{\theta}, \vec{\mu}) + C_{\text{local}} \ln a \right]. \quad (3.20)$$

We can evaluate this integral using saddle point method. In particular the saddle point of  $\chi$  integration is at the origin. Since near the origin  $\theta^a$ 's are degree one homogeneous functions of the coordinates  $\chi^m$ , the integration over the  $N_R$  variables  $\{\chi^m\}$  will give a factor of  $a^{-N_R(D-2)/2}$  according to (3.10). The integration over  $\beta$  gives a factor of  $a^{-(D-4)/2}$  and integration over the  $\mu_\alpha$ 's gives a factor of  $a^{-n_V(D-4)/2}$ . Combining these results we get

$$S_{\text{singlet}}(M, \vec{Q}) = S_{\text{BH}}(M, \vec{J} = 0, \vec{Q}) + \ln a \left( C_{\text{local}} - \frac{1}{2}(D-4) - \frac{1}{2}(D-2)N_R - \frac{1}{2}(D-4)n_V \right) + \dots. \quad (3.21)$$

## 4 Comparison with loop quantum gravity prediction

In loop quantum gravity there exist proposals for computing microscopic entropy of a Schwarzschild black hole [60–63]. These results give a formula for the degeneracy as a function of the eigenvalue of the area operator. In principle given an exact formula for the degeneracy one can extract its behaviour for large area by using asymptotic expansion formula. However there are different versions of this counting formula in loop quantum gravity, based on  $SU(2)$  Chern-Simons theory [60, 61] and  $U(1)$  Chern-Simons theories [62, 63]. We shall first review some of these results and then compare them with the result of the semi-classical analysis carried out in this paper.

The logarithmic correction to the black hole entropy based on the  $SU(2)$  Chern-Simons theory was first carried out in [29, 30], and justified more recently in [37–41]. The result for the entropy is given by

$$S^{(lqg)} = S_{\text{BH}} - 3 \ln a. \quad (4.1)$$

Before comparing this with our result there are three important points to consider:

1. First of all the analysis of [29, 30, 37–41] counted all states carrying a fixed number  $p_0$  of ‘punctures’. This number is an integer and is related to the area via the relation

$$p_0 \propto A_H. \quad (4.2)$$

Since  $A_H \sim M^2$  we have  $\delta p_0 \sim M \delta M$ . Now since  $p_0$  is integrally quantized the counting of [29, 30, 37–41] gives the number of microstates per unit  $p_0$  interval. This corresponds to  $\delta M \sim 1/M$ . Thus in order to get the number of states per unit mass interval, we need to multiply the number of states counted in [29, 30, 37–41] by  $M \sim a$ . This gives an additional contribution of  $\ln a$  to the entropy.

2. Since the presence of logarithmic correction to the effective action could change the equations of motion and hence the relation between mass and area, this could give additional logarithmic corrections to the entropy when we express the leading result in loop quantum gravity – given by a term proportional to the area – in terms of the mass of the black hole. This can certainly happen for a general charged black hole, but does not happen for a Schwarzschild black hole for the following reason. For Schwarzschild black hole the  $K_0(x)$  term differs from a multiple of the Gauss-Bonnet term by  $R_{\mu\nu}R^{\mu\nu}$  and  $R^2$  terms. Both these terms vanish on-shell, and furthermore their contribution to the equation of motion, being proportional to the first variation of these terms, also vanish on-shell. Finally the Gauss-Bonnet term being total derivative also does not contribute to the equations of motion. From this it follows that the Schwarzschild black hole solution does not receive any logarithmic correction. There can be corrections obtained by varying the  $\ln a$  term multiplying  $K_0(x)$  since  $a$  can be expressed in terms of the metric, but since  $\delta \ln a = \delta a/a$ , this will not be a logarithmic correction to the equations of motion. Thus we conclude that for the Schwarzschild black hole in  $D = 4$ , there is no additional logarithmic correction to the result (4.1) due to a modification of the relation between mass and area of the horizon.
3. [29, 30, 37–41] counts all spherically symmetric states, i.e. all states of zero angular momentum. Thus it gives the result for  $S_{\text{singlet}}$  described in §3.2.

The upshot of this discussion is that the loop quantum gravity prediction for  $S_{\text{singlet}}$  is given by adding to (4.1) a term  $\ln a$ . This gives

$$S_{\text{singlet}}^{(lqg)} = S_{\text{BH}} - 2 \ln a. \quad (4.3)$$

An alternative computation of the corrections to the Schwarzschild black hole entropy in loop quantum gravity, based on the U(1) Chern-Simons theory, has been given in [31, 34–36]. This also counts states with a fixed number of punctures and arrives at the result  $-\ln a$  for logarithmic correction to the entropy. The difference between this and the computation based

on SU(2) Chern-Simons theory can be traced to the different projections used in the two computations under the global part of the SU(2) Chern-Simons gauge group [38, 41]. While the analysis of [29, 30, 37–41] imposes the constraint that the states are singlets of the global part of the SU(2) gauge group, [31, 34–36] only requires invariance under a U(1) subgroup of this SU(2) group. If we identify the global part of the SU(2) group as the rotational isometry of the black hole [41], then the first one counts states with  $\sum_i J_i J_i = 0$  while the second one counts states with  $J_3 = 0$  but arbitrary  $\sum_i J_i J_i$ . In other words, the second result, after being converted to the number of states per unit mass range by adding  $\ln a$ , gives the entropy  $S_{\text{mc}}$  in loop quantum gravity:

$$S_{\text{mc}}^{(lqg)} = S_{\text{BH}}. \quad (4.4)$$

Note that there are no logarithmic corrections to  $S_{\text{mc}}$  in loop quantum gravity. This agrees with (4.3) since fixing  $J_3$  to 0 but letting  $\sum J_i J_i$  to be arbitrary produces an additive factor of  $2 \ln a$  in the entropy compared to the case where we fix both  $J_3$  and  $\sum J_i J_i$  to 0 (compare (3.12) and (3.21) for  $D = 4$ ,  $N_C = 1$ ,  $N_R = 3$ ).

Let us compare (4.3) with the prediction from Euclidean gravity analysis. Since we are considering a theory of pure gravity, we have, from (3.21), (2.35) with  $n_S = n_V = n_F = n_{3/2} = 0$ ,

$$S_{\text{singlet}} = S_{\text{BH}} + (C_{\text{local}} - 3) \ln a, \quad C_{\text{local}} = \frac{212}{45}. \quad (4.5)$$

This is different from (4.3), showing that the loop quantum gravity result for logarithmic correction to the entropy does not agree with the prediction of the Euclidean gravity analysis.

## 5 BTZ black holes

The logarithmic corrections to the entropy of BTZ black holes have been computed both from the microscopic [7] and macroscopic [8, 42, 43] perspective. In this section we shall see how the formalism described in this paper can be used to compute these corrections. This section does not contain any new results, but simply translates the existing analysis into the framework used in this paper for computing logarithmic corrections to the entropy. We shall first review the microscopic computation and then describe the macroscopic computation.

## 5.1 Microscopic computation

We shall consider a (1+1) dimensional conformal field theory with central charges  $(c, \bar{c})$ . If  $d_0(n, \bar{n})$  denotes the degeneracy of states carrying  $(L_0, \bar{L}_0)$  eigenvalues  $(n, \bar{n})$ , then we define the partition function as

$$Z(\tau, \bar{\tau}) = \text{Tr} \left[ e^{2\pi i \tau L_0 - 2\pi i \bar{\tau} \bar{L}_0} \right] = \sum_{n, \bar{n}} d_0(n, \bar{n}) e^{2\pi i n \tau - 2\pi i \bar{n} \bar{\tau}}. \quad (5.1)$$

In this sum  $n, \bar{n}$  are typically discrete but not necessarily integers, although  $n - \bar{n}$  takes integer values. For small  $\tau, \bar{\tau}$  the contribution to the integral comes from large  $n, \bar{n}$ . In this case we can approximate the sum by an integral of the form

$$Z(\tau, \bar{\tau}) \simeq \int dn \int d\bar{n} d(n, \bar{n}) e^{2\pi i n \tau - 2\pi i \bar{n} \bar{\tau}}, \quad (5.2)$$

where  $d(n, \bar{n})$  is some smooth function representing the average number of states per unit interval in  $n$  and  $\bar{n}$ . Note that  $d(n, \bar{n})$  could differ from  $d_0(n, \bar{n})$  by a large factor if the spacing between  $L_0 + \bar{L}_0$  eigenvalues is small. Now modular invariance of the theory implies that

$$Z(\tau, \bar{\tau}) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \text{Tr} \left[ e^{-2\pi i \frac{1}{\tau} L_0 + 2\pi i \frac{1}{\bar{\tau}} \bar{L}_0} \right]. \quad (5.3)$$

For small  $\tau, \bar{\tau}$  the contribution to the right hand side is dominated by the vacuum state with  $L_0 = -\frac{c}{24}$ ,  $\bar{L}_0 = -\frac{\bar{c}}{24}$  and we have

$$Z(\tau, \bar{\tau}) \simeq \exp \left[ \pi i \frac{c}{12\tau} - \pi i \frac{\bar{c}}{12\bar{\tau}} \right]. \quad (5.4)$$

This gives, for large  $n, \bar{n}$

$$d(n, \bar{n}) \simeq \int d^2\tau e^{-2\pi i n \tau + 2\pi i \bar{n} \bar{\tau}} Z(\tau, \bar{\tau}) \simeq \int d^2\tau e^{-2\pi i n \tau + 2\pi i \bar{n} \bar{\tau} + \pi i \frac{c}{12\tau} - \pi i \frac{\bar{c}}{12\bar{\tau}}}, \quad (5.5)$$

where it is understood that on the right hand side we pick the contribution to the integral from the saddle point close to the origin. Now the saddle point, obtained by extremizing the exponent on the right hand side of (5.5), is at

$$\tau_0 = i\sqrt{\frac{c}{24n}}, \quad \bar{\tau}_0 = -i\sqrt{\frac{\bar{c}}{24\bar{n}}}, \quad (5.6)$$

and the result of the integration is given by

$$\begin{aligned} d(n, \bar{n}) &\simeq \exp \left[ 2\pi\sqrt{\frac{cn}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{n}}{6}} \right] (-12\tau_0^3/ic)^{1/2} (12\bar{\tau}_0^3/i\bar{c})^{1/2} \\ &\simeq C_0 \exp \left[ 2\pi\sqrt{\frac{cn}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{n}}{6}} - \frac{3}{4} \ln n - \frac{3}{4} \ln \bar{n} \right] \end{aligned} \quad (5.7)$$

for some constant  $C_0$ .

## 5.2 Macroscopic computation

The three dimensional theory in the bulk that is dual to the  $CFT_2$  described in §5.1 is a theory of gravity in  $AdS_3$  space-time with Einstein-Hilbert term, cosmological constant term and gravitational Chern-Simons term, with the constants  $c$  and  $\bar{c}$  given by specific combinations of the parameters of the bulk theory:

$$c + \bar{c} = \frac{3l}{G}, \quad c - \bar{c} = 48\pi K, \quad (5.8)$$

where  $l$  is the radius of curvature of the dual  $AdS_3$  space,  $G$  is the Newton's constant and  $K$  is proportional to the coefficient of the gravitational Chern-Simons term.  $\ln d(n, \bar{n})$  has to be compared with the microcanonical entropy of a BTZ black hole of mass  $M$  and angular momentum  $J$  with the identification

$$Ml = n + \bar{n}, \quad J = n - \bar{n}. \quad (5.9)$$

Now the Bekenstein-Hawking entropy of a BTZ black hole [64] carrying mass  $M$  and angular momentum  $J$ , is given by (see *e.g.* [65])

$$S_{\text{BH}} = \pi \sqrt{\frac{c}{3}} (Ml + J) + \pi \sqrt{\frac{\bar{c}}{3}} (Ml - J) = 2\pi \sqrt{\frac{cn}{6}} + 2\pi \sqrt{\frac{\bar{c}\bar{n}}{6}}. \quad (5.10)$$

This is in perfect agreement with the leading terms in (5.7). Our goal will be to compute the logarithmic correction to the entropy from the macroscopic side. For this we need to first determine the scaling laws of various quantities with the size of the black hole. If  $a$  denotes the size of the black hole horizon then we have  $S_{\text{BH}} \sim a/G_N$ . Comparing this with (5.10) we see that here

$$M \sim a^2, \quad J \sim a^2, \quad S_{\text{BH}} \sim a. \quad (5.11)$$

Since we have a cosmological constant, it follows from the arguments at the end of §2 that the non-zero modes do not produce any logarithmic correction to the partition function. Furthermore none of the spatial translation generators commute with the generator of rotation in the two dimensional plane and hence there are no translational zero modes of the Euclidean black hole solution. As a result there are no logarithmic corrections to  $\ln Z$ , and the only logarithmic

corrections to the entropy come from the conversion of the grand canonical partition function to microcanonical entropy via the relation

$$Z(\beta, \omega) = \int dM \sum_J e^{S_{\text{mc}}(M, J) - \beta M - \omega J}, \quad (5.12)$$

or equivalently

$$e^{S_{\text{mc}}(M, J)} = \int d\beta d\omega Z(\beta, \omega) e^{\beta M + \omega J}. \quad (5.13)$$

It follows from (5.11) that we have  $\beta \sim a^{-1}$ ,  $\omega \sim a^{-1}$ ,  $\ln Z_{\text{cl}} \sim a$  and hence  $\partial^2 \ln Z_{\text{cl}} / \partial \beta^2 \sim \partial^2 \ln Z_{\text{cl}} / \partial \omega^2 \sim \partial^2 \ln Z_{\text{cl}} / \partial \beta \partial \omega \sim a^3$ . Thus integrations over  $\beta$  and  $\omega$  together produces a factor of  $a^{-3}$  and gives

$$e^{S_{\text{mc}}} \sim [Z e^{\beta M + \omega J}]_{\text{saddle}} a^{-3}. \quad (5.14)$$

Absence of logarithmic correction to  $Z$  now gives the logarithmic correction to  $S_{\text{mc}}$  to be

$$S_{\text{mc}} = S_{\text{BH}} - 3 \ln a. \quad (5.15)$$

### 5.3 Comparison of the microscopic and the macroscopic results

To compare the microscopic and the macroscopic results we note that (5.9) and (5.11) together gives

$$n \sim a^2, \quad \bar{n} \sim a^2. \quad (5.16)$$

Thus we can express the microscopic result (5.7) as

$$\ln d(n, \bar{n}) = S_{\text{BH}} - 3 \ln a. \quad (5.17)$$

This is in perfect agreement with the macroscopic result (5.15).

Note that both the macroscopic and the microscopic results (5.15) and (5.17) differ from the result  $-\frac{3}{2} \ln a$  given in [7] by a factor of 2. This can be traced to the fact that in [7] the entropy was calculated in a mixed ensemble in which the mass  $M = n + \bar{n}$  was fixed but the angular momentum  $J = n - \bar{n}$  was summed over. For this we need to set  $\tau + \bar{\tau} = 0$  and only integrate over  $\tau - \bar{\tau}$  in eq.(5.5). Evaluating the integral by saddle point method we shall get the result  $-\frac{3}{2} \ln a$ . Similarly on the macroscopic side the computation of the entropy in the mixed ensemble will involve a relation like (5.13) with  $S_{\text{mc}}(M, J)$  replaced by  $S_{\text{mixed}}(M)$  on the left hand side, and  $\int d\beta e^{\beta M} Z(\beta, \omega = 0)$  on the right hand side. The integration over  $\beta$  will now produce a factor of  $a^{-3/2}$ , giving us a logarithmic correction of  $-(3/2) \ln a$  to  $S_{\text{mixed}}$ .



Finally we note that the agreement between the logarithmic corrections in the microscopic and macroscopic analysis could also have been inferred by comparing the gravity partition function  $Z$  with the conformal field theory partition function  $Z(\tau, \bar{\tau})$  given in (5.4). The absence of logarithmic corrections to both the gravity partition function and the conformal field theory partition function is enough to ensure that the corresponding microcanonical entropies should also agree.

**Acknowledgement:** I would like to thank A. Ghosh, R. Kaul, Cynthia Keeler, P. Majumdar, K. Meissner and P. Mitra for useful communications. This work was supported in part by the J. C. Bose fellowship of the Department of Science and Technology, India and the project 11-R&D-HRI-5.02-0304.

## A Normalizability of the zero modes

A black hole solution breaks translation invariance, and, if it carries angular momentum, also breaks the rotational invariance to its Cartan subalgebra. Thus naively one would expect the solution to carry zero mode deformations associated with broken translational and rotational invariance. We shall however show that for the euclidean black hole only the zero modes associated with broken translation symmetry, which commute with the rotation generator  $\vec{\omega} \cdot \vec{T}$ , satisfy the required boundary conditions. All other zero modes are projected out.

Let us first analyze the zero modes of the Lorentzian black hole associated with broken translation invariance. These are generated by diffeomorphisms which approach constant translation at infinity and vanish as we approach the horizon. Since for testing normalizability we only need the behaviour of the deformation at infinity, we can work with constant translations. We shall begin by examining Schwarzschild solution in  $D$  dimensions. The metric takes the form [66]

$$ds^2 = -(1 - Cr^{-D+3})dt^2 + (1 - Cr^{-D+3})^{-1}dr^2 + r^2 d\Omega_{D-2}^2, \quad (\text{A.1})$$

where  $d\Omega_d$  is the line element on a unit  $d$ -sphere. If we introduce the Cartesian coordinates  $(x^0 = t, x^1, \dots, x^{D-1})$  in the usual manner and deform the solution by a diffeomorphism that approaches, for  $r \rightarrow \infty$ ,  $\delta x^i = a^i$  for some fixed  $(D-1)$  dimensional vector  $\vec{a}$ , then  $\eta_{\mu\nu} dx^\mu dx^\nu$  – the  $C$  independent part of the metric – remains invariant under this transformation. The change in the  $C$  dependent part can be computed using the fact that under this transformation

$\delta r = \vec{a} \cdot \vec{x}/r$ . Thus we have

$$\begin{aligned} \delta(ds^2) = & C(3-D)\vec{a} \cdot \vec{x} r^{1-D} dt^2 + (1 - Cr^{-D+3})^{-2} C(3-D)\vec{a} \cdot \vec{x} r^{1-D} dr^2 \\ & + 2 \left( (1 - Cr^{-D+3})^{-1} - 1 \right) (r^{-1} \vec{a} \cdot d\vec{x} - r^{-2} \vec{a} \cdot \vec{x} dr) dr, \end{aligned} \quad (\text{A.2})$$

to first order in the deformation parameter  $\vec{a}$ . This shows that in the asymptotically Cartesian coordinate system  $\delta g_{\mu\nu}$  is of order  $r^{2-D}$ . These are clearly normalizable deformations of the solution for  $D \geq 4$ . For rotating black holes the metric is more complicated but asymptotically the metric approaches the Minkowski metric at the same rate. Thus the translation zero modes are normalizable for these solutions as well.

Let us now analyze the fate of these zero modes when we consider Euclidean (rotating) black holes obtained by the replacement  $t \rightarrow -i\tau$  followed by periodic identification of the  $\tau$  coordinate. Let us for definiteness focus on the Kerr or Kerr-Newmann metric in four dimensions, but the analysis generalizes to higher dimensions. The metric near the horizon has a  $(d\phi + i\omega d\tau/\beta)^2$  factor which becomes singular at the horizon if we make the usual identification  $\tau \equiv \tau + \beta$  and  $\phi \equiv \phi + 2\pi$ . The remedy is to define  $\tilde{\phi} = \phi + i\omega\tau/\beta$  and make the identification  $\tilde{\phi} \equiv \tilde{\phi} + 2\pi$  and  $\tau \equiv \tau + \beta$ . In terms of the original  $(\phi, \tau)$  coordinate system this corresponds to an identification  $(\phi, \tau) \equiv (\phi - i\omega, \tau + \beta) \equiv (\phi + 2\pi, \tau)$ . Thus the question we need to address is whether the translation zero modes satisfy this periodicity restriction. From the description of the translation zero modes given above it is clear that under  $\tau \rightarrow \tau + \beta$  the parameters  $\vec{a}$  remain unchanged, whereas under  $\phi \rightarrow \phi - i\omega$ ,  $a_z \rightarrow a_z$ ,  $(a_x + ia_y) \rightarrow e^{\omega\beta}(a_x + ia_y)$ . Thus the zero mode generated by the parameter  $a_z$  satisfies the required boundary condition while the zero modes generated by  $a_x$  and  $a_y$  fail to satisfy the requirement of periodicity under  $(\phi, \tau) \rightarrow (\phi - i\omega, \tau + \beta)$ . For more general black holes in higher dimensions one can generalize this analysis to argue that only those translations which commute with the rotation generator  $\vec{\omega} \cdot \vec{T}$  generate zero modes. For  $\vec{\omega} = 0$ , i.e. for Schwarzschild black holes, all the translation generators produce zero modes of the Euclidean black hole solution.

We now turn to the rotational zero modes. In this case asymptotically the black hole metric deviates from the rotationally invariant metric by a term of order  $dt dx^i / r^{D-2}$ . On the other hand under a rotation  $\delta x^i \sim r$ . Thus the asymptotic form of  $\delta g_{ti}$  associated with a rotational zero mode is of order  $1/r^{D-2}$  and again this describes a normalizable deformation for  $D \geq 4$ . As a result for Lorentzian rotating black holes there are zero modes for every broken rotation generator. However things are again different in the Euclidean theory. Following the logic of the previous two subsections we can show that all the rotational zero modes transform

non-trivially under  $\vec{\omega}.\vec{T}$  and hence fail to satisfy the periodicity requirement imposed by the Euclidean solution. Thus for the euclidean black hole there are no rotational zero modes.

## B Subtraction of the thermal gas contribution to the partition function

In this appendix we shall identify the thermal gas contribution to the black hole partition function and argue that the subtraction scheme used in the text, in which we divide the partition function  $Z$  of the original black hole of size  $a$  and confined in a box of size  $L$  by the partition function  $Z_0$  of a black hole of fixed size  $a_0$  and confined in a box of size  $La_0/a$ , correctly removes not only the contribution to  $Z$  from the thermal gas but also the spurious boundary contributions.

In order to gain some insight into the problem, it will be useful to recall how the euclidean partition function in the background of a flat space-time in which the euclidean time coordinate has period  $\beta$  sees the thermal contribution to the partition function. In this case the eigenvalues of the kinetic operator have the form:

$$\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2, \quad (\text{B.1})$$

with density of states

$$d\mu = \frac{V}{(2\pi)^{D-1}} d^{D-1}k + \dots, \quad (\text{B.2})$$

where  $V \sim L^{D-1}$  is the total volume of the box inside which we restrict the spatial coordinates and  $\dots$  denote subleading corrections involving lower powers of  $L$  which arise from boundary effects. Thus the net contribution to the partition function – which we shall denote by  $Z_{\text{free}}$  – is given by

$$\ln Z_{\text{free}}(\beta, L, \epsilon) = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \sum_{n=-\infty}^{\infty} \int d\mu e^{-4\pi^2 n^2 s / \beta^2 - s \vec{k}^2}. \quad (\text{B.3})$$

We now perform the sum over  $n$  by Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \tilde{f}(m), \quad \tilde{f}(m) = \int dx e^{-2\pi i m x} f(x), \quad (\text{B.4})$$

to express (B.3) as

$$\ln Z_{\text{free}} = \frac{1}{2} \int_{\epsilon}^{\infty} \frac{ds}{s} \sqrt{\frac{\beta^2}{4\pi s}} \sum_{m=-\infty}^{\infty} \int d\mu e^{-m^2 \beta^2 / 4s - s \vec{k}^2}. \quad (\text{B.5})$$

First consider the contribution from the  $V d^{D-1}k/(2\pi)^{D-1}$  term in  $d\mu$ . The  $m = 0$  term is ultraviolet divergent, but it is independent of  $\beta$  except for an overall multiplicative factor of  $\beta$ . This has the interpretation of one loop contribution to the cosmological constant and will be cancelled by a local counterterm if the final theory does not have a cosmological constant. The  $m \neq 0$  terms have their integrand suppressed exponentially for  $s \ll a^2$  and hence are free from ultraviolet divergences. For these terms we can set  $\epsilon = 0$  and it is easy to see that the result will be proportional to  $V\beta^{-D+1}$ . This is precisely the thermal contribution to the partition function.

Next consider the effect of the boundary corrections to the density of states, represented by  $\dots$  in (B.2). It will be important to understand the nature of these contributions since the analog of such contributions for the black hole problem must also be removed from the partition function if we are to compute the partition function associated with black hole microstates. Like the bulk contribution, the  $m = 0$  term may be ultraviolet divergent, i.e. divergent in the  $\epsilon \rightarrow 0$  limit. Since the  $\epsilon$  dependence of the contribution can be inferred from the small  $s$  expansion of the heat kernel, which naturally splits into a bulk and the boundary contribution, we can remove the ultraviolet divergent part of the boundary contribution by simply removing from the heat kernel the boundary contribution to the non-positive powers of  $s$  in the small  $s$  expansion. As explained in the second point in the discussion below (2.22), this is the prescription used in §2 for dealing with the boundary contribution to the heat kernel. The rest of the contribution, being  $\epsilon$  independent, now depends only on  $L$  and  $\beta$ . By dimensional analysis this must depend on  $L/\beta$ . Thus if we subtract from this  $\ln Z_{\text{free}}(\beta_0, L_0, \epsilon)$  with  $L_0 = L\beta_0/\beta$  then the boundary contributions will cancel altogether irrespective of their form. (In this case the bulk contributions, representing the thermal partition function, also cancel completely, but this will not be the case for the black hole.)

Let us now consider the effect of the presence of the black hole inside the box. In this case we do not have a factorization between the contribution from the modes along the time circle and the modes along the spatial directions. As a result the eigenvalues of the kinetic operator have more complicated form than (B.1), and the density of states depends on the quantum number  $n$  labelling momentum along the Euclidean time circle. Also since for large distance away from the black hole the spatial part of the kinetic operator resembles the Hamiltonian of a charged particle in a Coulomb field, the eigenfunctions are not plane waves but have additional phases which depend on the radial coordinate. Nevertheless our general arguments hold. In particular in the large  $L$  limit the dominant contribution to  $\ln Z$ , proportional to  $L^{D-1}$ , is expected to

be identical to that for a thermal gas without a black hole, and is cancelled against a similar factor in the expression for  $\ln Z_0$ . For the terms containing subleading powers of  $L$ , once we remove from the short distance expansion of the heat kernel the boundary terms containing non-positive powers of  $s$ , there is no ultraviolet divergent contribution associated with the boundary and the  $L$ -dependent contribution depends only on the ratio  $L/a$ . Since  $L_0/a_0 = L/a$ , subtracting  $\ln Z_0$  from  $\ln Z$  also removes all possible subleading boundary contributions which depend on  $L/a$ . This is reflected in the fact that the contribution (2.19) to  $\ln(Z/Z_0)$  can be evaluated without any knowledge of the box in which we confine the black hole, as long as we use the prescription of dropping the non-positive powers of  $s$  in the boundary contribution to the heat kernel.

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